Example 1. Find the particular solution of the system

$$x' = 4x - 3y, \quad y' = 6x - 7y.$$

$$x(0) = 2, \quad y(0) = -1$$

$y' = 6x - 7y$, then $x = \frac{1}{6}y' + \frac{7}{6}y$

so that $x' = \frac{1}{6}y'' + \frac{7}{6}y'$.

Substituting into $x' = 4x - 3y$, we get

$$\frac{1}{6}y'' + \frac{7}{6}y' = 4\left(\frac{1}{6}y' + \frac{7}{6}y\right) - 3y$$

or

$$y'' + 3y' - 10y = 0. \quad \text{(roots of } -2, 5)$$

Thus $y = c_1 e^{2t} + c_2 e^{-5t}$

Therefore $x = \frac{1}{6}\left(2c_1 e^{2t} - 5c_2 e^{-5t}\right) + \frac{7}{6}\left(c_1 e^{2t} + c_2 e^{-5t}\right)$

$$= \frac{3}{2}c_1 e^{2t} + \frac{1}{3}c_2 e^{-5t}.$$

Using initial conditions, $\left.\begin{array}{l} x(0) = \frac{3}{2}c_1 + \frac{1}{3}c_2 = 2 \\ y(0) = c_1 + c_2 = -1 \end{array}\right\} \begin{array}{l} c_1 = 2 \\ c_2 = -3. \end{array}$

Thus $x(t) = 3e^{2t} - e^{-5t}, \quad y(t) = 2e^{2t} - 3e^{-5t}.$

Example 2. Find a general solution of the system

$$\begin{aligned}(x' - 4x) + 3y &= 0,\\ -6x + (y' + 7y) &= 0.\end{aligned}$$

We view D as the derivative operator. Then rewrite as

$$\begin{aligned}(D-4)x + 3y &= 0\\ -6x + (D+7)y &= 0.\end{aligned}$$

Take $\det\begin{pmatrix} D-4 & 3 \\ -6 & D+7 \end{pmatrix} = (D-4)(D+7) - 3(-6) = D^2+3D-10.$

Write solution vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.

Then $\det\begin{pmatrix} 0 & 3 \\ 0 & D+7 \end{pmatrix} = 0$ gives $(D^2+3D-10)x = 0$

and

$\det\begin{pmatrix} D-4 & 0 \\ -6 & 0 \end{pmatrix} = 0$ gives $(D^2+3D-10)y = 0.$

Or

$$\left.\begin{aligned} x''+3x'-10x=0 \\ y''+3y'-10y=0 \end{aligned}\right\} \quad \begin{aligned} x &= a_1e^{2t} + a_2e^{-5t} \\ y &= b_1e^{2t} + b_2e^{-5t}. \end{aligned}$$

Homework. 1-11 (odd)